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Periodic solutions for p-Laplacian neutral functional differential equation with deviating arguments ☆

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Abstract

By using the theory of coincidence degree, we study a kind of periodic solutions to p-Laplacian neutral functional differential equation with deviating arguments such as $(\varphi_p(x(t) - cx(t-\sigma))')' + g(t, x(t-\tau(t))) = p(t)$, a result on the existence of periodic solutions is obtained.

Keywords: Deviating argument; Periodic solution; Theory of coincidence degree

1. Introduction

The problem of periodic solutions of ordinary differential equation was extensively studied, see Refs. [1–4]. In recent years, there are many results about periodic solutions to second-order scalar differential equations with deviating arguments [5–10]. For example, in [10] the authors studied the following equation with a deviating argument:

$$x''(t) + f\big(x(t)\big)x'(t) + g\Big(x\big(t - \tau\big(t, x(t)\big)\big)\Big) = e(t). \tag{1.1}$$

By using Mawhin's continuation theorem, some results on the existence of periodic solution are obtained. But the corresponding problem of p-Laplacian differential equation with deviating arguments has been studied far less often. The reason for this is that the differential operator

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$(\varphi_p(u))' = (|u|^{p-2}u)'$ $(p \neq 2)$ is no longer linear, so the theory of coincidence degree cannot be applied directly. In [11], we found that the authors studied periodic solutions for the following p-Laplacian Liénard equation with a deviating argument:

$$\big(\varphi_p\big(x'(t)\big)\big)' + f\big(t, x(t)\big)x'(t) + \beta(t)g\big(x\big(t - \tau(t)\big)\big) = e(t). \tag{1.2}$$

The condition imposed on $g(x)$ is either $\lim_{|x|\to+\infty} \sup |\frac{g(x)}{x}| < r$ or $|g(u) - g(v)| \leqslant l|u - v|$.

In this paper, we study the existence of periodic solutions for p-Laplacian neutral functional differential equation with deviating arguments

$$\big(\varphi_p\big(x(t) - cx(t - \sigma)\big)'\big)' + g\big(t, x\big(t - \tau(t)\big)\big) = p(t), \tag{1.3}$$

where $\varphi_p : R \to R$, $\varphi_p(u) = |u|^{p-2}u$, $g \in C(R^2, R)$, $\tau(t)$, $p(t)$ are continuous periodic functions defined on R with period $T > 0$, $\sigma, c \in R$ are constants such that $|c| \neq 1$. By using the theory of coincidence degree, we obtain a new result to guarantee the existence of periodic solutions. The significance is that the condition

$$\lim_{x\to-\infty} \sup_{t\in[0,T]} \frac{|g(t, x) - p(t)|}{|x|^{p-1}} \leqslant r$$

imposed on $g(t, x)$ is very weak and we only need $|c| \neq 1$. Furthermore, an example is given to demonstrate our result.

2. Main lemmas

We first rewrite Eq. (1.3) in the following form:

$$\begin{cases} (Ax_1)'(t) = \varphi_q(x_2(t)) = |x_2(t)|^{q-2}x_2(t), \\ x_2'(t) = -g(t, x_1(t - \tau(t))) + p(t), \end{cases} \tag{2.1}$$

where $1/p + 1/q = 1$. We can easily see if $x(t) = (x_1(t), x_2(t))^T$ is a T-periodic solution of Eq. (2.1), then $x_1(t)$ is a T-periodic solution of Eq. (1.3).

We set the following notations: $T > 0$ is a constant, $C_T = \{\varphi \in C(R, R)\colon\ \varphi(t + T) \equiv \varphi(t)\}$ with the norm $|\varphi|_0 = \max_{t\in[0,T]} |\varphi(t)|$, $X = Y = \{x = (x_1(\cdot), x_2(\cdot))^T \in C(R, R^2)\colon\ x(t) \equiv x(t + T)\}$ with the norm $\|x\| = \max\{|x_1|_0, |x_2|_0\}$, $|x|_p = \int_0^T |x(t)|^p\,dt$. Clearly, X and Y are *Banach* spaces. We also defined operators A and L in the following form:

$$A : C_T \to C_T, \quad (Ax)(t) = x(t) - cx(t - \sigma),$$

$$L : D(L) \subset C_T \to C_T, \quad Lx = \begin{pmatrix} (Ax_1)' \\ x_2' \end{pmatrix},$$

where $x = \binom{x_1}{x_2}$.

Lemma 2.1. [12] *If $|c| \neq 1$, then A has continuous bounded inverse on C_T, and*

(1) $\|A^{-1}x\| \leqslant \frac{\|x\|}{||c|-1|}$, $\forall x \in C_T$;
(2) $\int_0^T |(A^{-1}f)(t)|\,dt \leqslant \frac{1}{|1-|c||} \int_0^T |f(s)|\,ds$, $\forall f \in C_T$;
(3) $\int_0^T |(A^{-1}f)(t)|^2\,dt \leqslant \frac{1}{(1-|c|)^2} \int_0^T |f(s)|^2\,ds$, $\forall f \in C_T$.

By Hale's terminology [13], a solution $x(t)$ of Eq. (1.3) is that $x \in C(R, R)$ such that $Ax \in C^1(R, R)$ and Eq. (1.3) is satisfied on R. In general, x is not $C^1(R, R)$. But from Lemma 2.1, it is easy to see that $(Ax)' = Ax'$. So a T-periodic solution x of Eq. (1.3) must be $C^1(R, R)$. According to the first part of Lemma 2.1, we can easily obtain that $\ker L = R^2$, $\operatorname{Im} L = \{y \in Y: \int_0^T y(s)\,ds = 0\}$. So L is a *Fredholm* operator with index zero. Let project operators P, Q be as follows:

$$P: X \to \ker L, \quad Px = \frac{1}{T}\int_0^T x(s)\,ds; \qquad Q: Y \to \operatorname{Im} Q \subset R^2, \quad Qy = \frac{1}{T}\int_0^T y(s)\,ds,$$

then $\operatorname{Im} P = \ker L$, $\ker Q = \operatorname{Im} L$. Set $L_p = L|_{D(L)\cap \ker P}$ and $L_p^{-1}: \operatorname{Im} L \to D(L)$ denotes the inverse of L_p, then

$$\left[L_p^{-1} y\right](t) = \begin{pmatrix} (A^{-1} F y_1)(t) \\ (F y_2)(t) \end{pmatrix},$$

$$[Fy](t) = -\int_t^T y(s)\,ds + \int_0^T \frac{s}{T} y(s)\,ds, \tag{2.2}$$

where $y(t) = \binom{y_1(t)}{y_2(t)}$.

Lemma 2.2. [13] *Let X and Y be two Banach spaces, $L: D(L) \subset X \to Y$ be a Fredholm operator with index zero, $\Omega \subset X$ be an open bounded set, and $N: \overline{\Omega} \to Y$ be L-compact on $\overline{\Omega}$. If all the following conditions hold:*

(1) *$Lx \neq \lambda Nx$, $\forall x \in \partial\Omega \cap D(L)$, $\forall \lambda \in (0, 1)$;*
(2) *$Nx \notin \operatorname{Im} L$, $\forall x \in \partial\Omega \cap \operatorname{Ker} L$;*
(3) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$, *where $J: \operatorname{Im} Q \to \ker L$ is an isomorphism,*

then equation $Lx = Nx$ has a solution on $\Omega \cap D(L)$.

3. Main result

Theorem. *Suppose that $p > 2$ and there exist positive constants D and $r \geqslant 0$ such that*

$[H_1]$ $x[g(t, x) - p(t)] > 0, \forall t \in R, |x| > D$;
$[H_2]$ $\lim_{x\to-\infty} \sup_{t\in[0,T]} \frac{|g(t,x)-p(t)|}{|x|^{p-1}} \leqslant r$.

Then Eq. (1.2) *has at least one T-periodic solution, if* $\frac{2T^p r(1+|c|)}{(1-|c|)^2} < 1$.

Proof. We easily see that Eq. (2.1) has a T-periodic solution if and only if the following operator equation

$$Lx = Nx,$$

has a T-periodic solution, where $N: C_T \to C_T$,

$$(Nx)(t)=\begin{pmatrix}\varphi_q(x_2(t))\\ -g(t,x_1(t-\tau(t)))+p(t)\end{pmatrix}.$$

From (2.2), we see that N is L-compact on $\overline{\Omega}$, where Ω is any open, bounded subset of C_T. Take

$$\Omega_1=\big\{x\colon\ x\in C_T,\ Lx=\lambda Nx,\ \lambda\in[0,T]\big\}.$$

$\forall x=\binom{x_1(t)}{x_2(t)}\in\Omega_1$, then x must satisfy

$$\begin{cases}(Ax_1)'(t)=\lambda\varphi_q(x_2(t))=\lambda|x_2(t)|^{q-2}x_2(t),\\ x_2'(t)=-\lambda g(t,x_1(t-\tau(t)))+\lambda p(t).\end{cases}\tag{3.1}$$

From the first equation of (3.1), we can know $x_2(t)=\varphi_p(\frac{1}{\lambda}(Ax_1)'(t))$, which together with the second equation of (3.1) yields

$$\left(\varphi_p\left(\frac{1}{\lambda}(Ax_1)'(t)\right)\right)'+\lambda g\big(t,x_1\big(t-\tau(t)\big)\big)=\lambda p(t),\quad\text{i.e.,}$$

$$\big(\varphi_p\big((Ax_1)'(t)\big)\big)'+\lambda^p g\big(t,x_1\big(t-\tau(t)\big)\big)=\lambda^p p(t).\tag{3.2}$$

Integrating both sides of (3.2) over $[0,T]$, we have

$$\int_0^T\big[g\big(t,x_1\big(t-\tau(t)\big)\big)-p(t)\big]\,dt=0.\tag{3.3}$$

By integral mean value theorem, there is a constant $\xi\in[0,T]$ such that $g(\xi,x_1(\xi-\tau(\xi)))-p(\xi)=0$. So from assumption $[H_1]$ we can get $|x_1(\xi-\tau(\xi))|\leqslant D$. So

$$|x_1|_0\leqslant D+\int_0^T\big|x_1'(t)\big|\,dt.\tag{3.4}$$

On the other hand, multiplying the two sides of Eq. (3.2) by $(Ax_1)(t)$ and integrating them over $[0,T]$, we get

$$\begin{aligned}-\int_0^T\big|(Ax_1)'(t)\big|^p\,dt&=-\int_0^T\big|Ax_1'(t)\big|^p\,dt\\ &=-\lambda^p\int_0^T\big[g\big(t,x_1\big(t-\tau(t)\big)\big)-p(t)\big]\big[x_1(t)-cx_1(t-\sigma)\big]\,dt,\end{aligned}\tag{3.5}$$

i.e.,

$$\begin{aligned}\int_0^T\big|\big(Ax_1'\big)(t)\big|^p\,dt&=\lambda^p\int_0^T\big[g\big(t,x_1\big(t-\tau(t)\big)\big)-p(t)\big]\big[x_1(t)-cx_1(t-\sigma)\big]\,dt\\ &\leqslant|x_1|_0\int_0^T\big|g\big(t,x_1\big(t-\tau(t)\big)\big)-p(t)\big|\,dt\end{aligned}$$

$$+ |c| \int_0^T \big|g\big(t, x_1\big(t-\tau(t)\big)\big) - p(t)\big|\big|x_1(t-\sigma)\big|\,dt$$

$$\leqslant \big(1+|c|\big)|x_1|_0 \int_0^T \big|g\big(t, x_1\big(t-\tau(t)\big)\big) - p(t)\big|\,dt. \tag{3.6}$$

In view of $\frac{2T^p r(1+|c|)}{(1-|c|)^2} < 1$, there exists a constant $\varepsilon > 0$ such that $\frac{2T^p(1+|c|)(r+\varepsilon)}{(1-|c|)^2} < 1$.

From assumption $[H_2]$, we get that there exists a constant $\rho > 0$ such that

$$\big|g(t,x) - p(t)\big| \leqslant (r+\varepsilon)|x|^{p-1}, \quad \forall t \in R,\ x < -\rho. \tag{3.7}$$

Let $E_1 = \{t \in [0,T]\colon\ x_1(t-\tau(t)) < -\rho\}$, $E_2 = \{t \in [0,T]\colon\ |x_1(t-\tau(t))| \leqslant \rho\}$, $E_3 = \{t \in [0,T]\colon x_1(t-\tau(t)) > \rho\}$. By the second equation of (3.1) it is easy to see that

$$\bigg(\int_{E_1} + \int_{E_2} + \int_{E_3}\bigg)\big[g\big(t, x_1\big(t-\tau(t)\big)\big) - p(t)\big]\,dt = 0. \tag{3.8}$$

Hence

$$\begin{aligned}
\int_{E_3} \big|g\big(t, x_1\big(t-\tau(t)\big)\big) - p(t)\big|\,dt &= \int_{E_3} \big[g\big(t, x_1\big(t-\tau(t)\big)\big) - p(t)\big]\,dt \\
&= -\bigg(\int_{E_1} + \int_{E_2}\bigg)\big[g\big(t, x_1\big(t-\tau(t)\big)\big) - p(t)\big]\,dt \\
&\leqslant \bigg(\int_{E_1} + \int_{E_2}\bigg)\big|g\big(t, x_1\big(t-\tau(t)\big)\big) - p(t)\big|\,dt.
\end{aligned} \tag{3.9}$$

Therefore, by (3.7) and (3.9) we get

$$\begin{aligned}
\int_0^T \big|g\big(t, x_1\big(t-\tau(t)\big)\big) - p(t)\big|\,dt &= \bigg(\int_{E_1} + \int_{E_2} + \int_{E_3}\bigg)\big|g\big(t, x_1\big(t-\tau(t)\big)\big) - p(t)\big|\,dt \\
&\leqslant 2\bigg(\int_{E_1} + \int_{E_2}\bigg)\big|g\big(t, x_1\big(t-\tau(t)\big)\big) - p(t)\big|\,dt \\
&\leqslant 2\int_{E_1} (r+\varepsilon)\big|x_1\big(t-\tau(t)\big)\big|^{p-1}\,dt + 2\tilde{g}_\rho T \\
&\leqslant 2(r+\varepsilon)T|x_1|_0^{p-1} + 2\tilde{g}_\rho T,
\end{aligned} \tag{3.10}$$

where $\tilde{g}_\rho = \max_{t \in E_2} |g(t, x_1(t-\tau(t))) - p(t)|$. From (3.6) and (3.10), we know

$$\int_0^T \big|\big(Ax_1'\big)(t)\big|^p\,dt \leqslant 2\big(1+|c|\big)T\big[(r+\varepsilon)|x_1|_0^p + \tilde{g}_\rho|x|_0\big]. \tag{3.11}$$

Substituting (3.4) into (3.11), we have

$$\int_0^T \left|\left(Ax_1'\right)(t)\right|^p dt \leqslant 2\left(1+|c|\right)T(r+\varepsilon)\left(D+\int_0^T \left|x_1'(t)\right| dt\right)^p$$
$$+2\left(1+|c|\right)T\tilde{g}_\rho\left(D+\int_0^T \left|x_1'(t)\right| dt\right). \tag{3.12}$$

Case (1). If $\int_0^T |x_1'(t)|\,dt = 0$, from (3.4) we see $|x_1|_0 < D$.
Case (2). If $\int_0^T |x_1'(t)|\,dt > 0$, then we know

$$\left(D+\int_0^T \left|x_1'(t)\right| dt\right)^p = \left(\int_0^T \left|x_1'(t)\right| dt\right)^p \left(1+\frac{D}{\int_0^T |x_1'(t)|\,dt}\right)^p. \tag{3.13}$$

By the knowledge of mathematical analysis, there is a constant $\delta > 0$ such that

$$(1+x)^p < 1+(1+p)x, \quad \forall x \in [0,\delta]. \tag{3.14}$$

If $D/\int_0^T |x_1'(t)|\,dt > \delta$, then $\int_0^T |x_1'(t)|\,dt < D/\delta$, so from (3.4) we have $|x_1|_0 < D/\delta + D$.

If $D/\int_0^T |x_1'(t)|\,dt \leqslant \delta$, then by (3.14) we know

$$\begin{aligned}\left(D+\int_0^T \left|x_1'(t)\right| dt\right)^p &\leqslant \left(\int_0^T \left|x_1'(t)\right| dt\right)^p \left(1+\frac{(p+1)D}{\int_0^T |x_1'(t)|\,dt}\right)\\ &= \left(\int_0^T \left|x_1'(t)\right| dt\right)^p + (p+1)D\left(\int_0^T \left|x_1'(t)\right| dt\right)^{p-1}\\ &\leqslant T^{p/q}\int_0^T \left|x_1'(t)\right|^p dt + (p+1)DT^{\frac{p-1}{q}}\left(\int_0^T \left|x_1'(t)\right|^p dt\right)^{1/q}.\end{aligned}$$

By (3.12) we obtain

$$\begin{aligned}\int_0^T \left|Ax_1'(t)\right|^2 dt &\leqslant T^{\frac{p-2}{p}}\left[\int_0^T \left|Ax_1'(t)\right|^p dt\right]^{2/p} \leqslant T^{\frac{p-2}{p}}\int_0^T \left|Ax_1'(t)\right|^p dt\\ &\leqslant 2T^p\left(1+|c|\right)(r+\varepsilon)\int_0^T \left|x_1'(t)\right|^p dt\\ &\quad + 2T^{\frac{pq-1}{q}}\left(1+|c|\right)(p+1)D(r+\varepsilon)\left(\int_0^T \left|x_1'(t)\right|^p dt\right)^{1/q}\\ &\quad + 2T\left(1+|c|\right)\tilde{g}_\rho\left[D+T^{1/q}\left(\int_0^T \left|x_1'(t)\right|^p dt\right)^{1/p}\right].\end{aligned} \tag{3.15}$$

By applying the third part of Lemma 2.1, we get

$$\int_0^T \left|x_1'(t)\right|^2 dt = \int_0^T \left|\left(A^{-1}Ax_1'\right)(t)\right|^2 dt \leqslant \frac{\int_0^T |(Ax_1')(t)|^2\,dt}{(1-|c|)^2}.$$

So it follows from (3.15) that

$$\begin{aligned}\int_0^T \left|x_1'(t)\right|^2 dt &\leqslant \frac{2T^p(1+|c|)(r+\varepsilon)}{(1-|c|)^2}\int_0^T \left|x_1'(t)\right|^p dt \\ &\quad + \frac{2(1+|c|)(p+1)D(r+\varepsilon)}{(1-|c|)^2}T^{\frac{pq-1}{q}}\left(\int_0^T \left|x_1'(t)\right|^p dt\right)^{1/q} \\ &\quad + \frac{2T(1+|c|)\tilde{g}_\rho}{(1-|c|)^2}\left[D + T^{1/q}\left(\int_0^T \left|x_1'(t)\right|^p dt\right)^{1/p}\right].\end{aligned}$$

As $q > 1$, $\frac{2T^p(1+|c|)(r+\varepsilon)}{(1-|c|)^2} < 1$, there is a constant $M_1 > 0$ such that $\int_0^T |x_1'(t)|^2\,dt \leqslant M_1$. It follows from (3.4) that

$$|x_1|_0 \leqslant D + T^{1/2}M_1^{1/2} := M_2.$$

By the first equation of (3.1) we have $\int_0^T |x_2(t)|^{q-2}x_2(t)\,dt = 0$, which implies there is a constant $t_1 \in [0,T]$ such that $x_2(t_1) = 0$. So $|x_2|_0 \leqslant \int_0^T |x_2'(t)|\,dt$. By the second equation of (3.1) we obtain

$$\int_0^T \left|x_2'(t)\right| dt \leqslant \int_0^T \left|g\big(t, x_1\big(t-\tau(t)\big)\big)\right| dt + \int_0^T \left|p(t)\right| dt \leqslant Tg_{M_2} + |p|_1,$$

where $g_{M_2} = \max_{|x|\leqslant M_2,\, t\in[0,T]} |g(t,x)|$. So we have

$$|x_2|_0 \leqslant Tg_{M_2} + |p|_1 := M_3.$$

Let $M = \sqrt{M_2^2 + M_3^2} + 1$, $\Omega = \{x = (x_1,x_2)^T\colon\ |x_1|_0 < M,\ |x_2|_0 < M\}$ and $\Omega_2 = \{x \in \partial\Omega\colon x \in \ker L\}$, then

$$QNx = \frac{1}{T}\int_0^T \begin{pmatrix} \varphi_q(x_2(t)) \\ -g(t, x_1(t-\tau(t))) + p(t)\end{pmatrix} dt = \begin{pmatrix} |x_2|^{q-2}x_2 \\ -g(t,x_1) + p(t)\end{pmatrix}.$$

If $QNx = 0$, then $x_2 = 0$, $x_1 = M$ or $-M$. But when $x_1 = M$, we know $-g(t,x_1) + p(t) < 0$, which yields a contradiction. Similarly when $x_1 = -M$, we also have $QNx \neq 0$, i.e., $\forall x \in \Omega$, $x \notin \operatorname{Im} L$. So conditions (1) and (2) of Lemma 2.2 are both satisfied. Next we show that condition (3) of Lemma 2.2 is also satisfied. Define the isomorphism $J : \operatorname{Im} Q \to \ker L$ as follows:

$$J(x_1,x_2)^T = (x_2,x_1)^T.$$

Let $H(\mu,x) = \mu x + \frac{1-\mu}{T}JQNx$, $(\mu,x) \in \Omega \times [0,1]$, then we have

$$H(\mu,x)=\begin{pmatrix}\mu x_1+\frac{1-\mu}{T}\left(\frac{1}{T}\int_0^T[-g(t,x_1)+p(t)]\,dt\right)\\ (\mu+\frac{1-\mu}{T}|x_2|^{q-2})x_2\end{pmatrix},$$
$$\forall(x,\mu)\in(\partial\Omega\cap\ker L)\times[0,1].$$

If $H(\mu,x)=0$, then $x_2=0$, $x_1=M$ or $-M$. Similar to the above proof we can see that $H(\mu,x)\neq 0$. Hence

$$\deg\{JQN,\Omega\cap\ker L,0\}=\deg\bigl\{H(0,x),\Omega\cap\ker L,0\bigr\}=\deg\bigl\{H(1,x),\Omega\cap\ker L,0\bigr\}$$
$$=\deg\{I,\Omega\cap\ker L,0\}\neq 0.$$

So condition (3) of Lemma 2.2 is satisfied. By applying Lemma 2.2, we conclude that equation $Lx=Nx$ has a solution $x(t)=(x_1(t),x_2(t))^T$ on $\overline{\Omega}\cap D(L)$, i.e., Eq. (1.3) has a T-periodic solution $x_1(t)$. □

Corollary. *Suppose that $p>2$ and there exist positive constants D and r such that*

$[H_1^*]$ $x[g(t,x)-p(t)]>0$, $\forall t\in R$, $|x|>D$;
$[H_2^*]$ $\lim_{x\to+\infty}\sup_{t\in[0,T]}|\frac{g(t,x)-p(t)}{x}|\leqslant r$.

Then Eq. (1.3) *has at least one T-periodic solution, if* $\frac{2T^p r(1+|c|)}{(1-|c|)^p}<1$.

As an application,we consider the following equation

$$\varphi_3\bigl(x(t)-5x(t-\pi)'\bigr)'+g\bigl(t,x(t-\sin t)\bigr)=e^{\cos^2 t},\tag{3.16}$$

where

$$g(t,x)=\begin{cases}e^{\sin^2 t}x^7, & x\geqslant 0,\\ \frac{x}{18e\pi^2}e^{\sin^2 t}, & x<0.\end{cases}$$

According to the theorem, we have $p=3$, $c=5$, $r=\frac{1}{18\pi^2}$, so $\frac{2T^p r(1+|c|)}{(1-|c|)^2}<1$. Hence, by using our theorem we know Eq. (3.16) has at least one 2π-periodic solution.

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